# Birational Complexity:

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**Definition:** The "complexity" of a log pair $(X, B)$ is

$$c(X,B) = \dim X + \dim \mathrm{Cl}_{\mathbb{Q}}(X) - |B|,$$

where $|B|$ is the sum of the coeff of $B$.

**Theorem (Brown–McKernan–Svaldi–Zong, 2016)** Let $(X,B)$ be a log CY pair. Then $c(X,B) \geq 0$ and if $c(X,B) < 1$ then $(X, \lfloor B \rfloor)$ is toric.

**Definition:** Let $(X,B)$ be a log pair. We say that $(Y, B_Y)$ is a "crepant model" if there is $\varphi : Y \dashrightarrow X$ birational, and a common $p: Z \to Y$ & $q: Z \to X$ for which

log pair $(B_Y \geq 0)$.

$$q^*(K_X + B) = p^*(K_Y + B_Y).$$

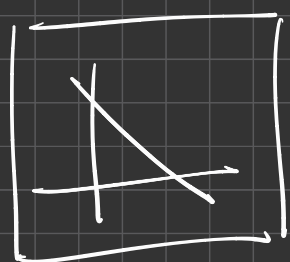
Notation: $(X,B) \simeq_{bir} (Y, B_Y)$.

**Definition:** The "birational complexity" of $(X, B)$ is defined to be:

$$c_{bir}(X, B) = \inf \left\{ c(Y, B_Y) \;\middle|\; \begin{array}{c} (Y, B_Y) \simeq_{bir} (X, B) \\ B_Y \geq 0. \end{array} \right\}$$

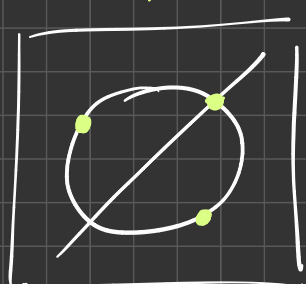
$c_{bir}(X, B) \geq 0$ provided that $(X, B)$ is log CY.

**Examples:**

$\mathbb{P}^2 =$ 

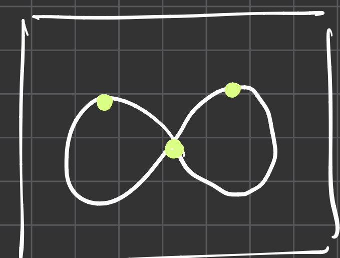
$c(\mathbb{P}^2, B_0) = 2 + 1 - 3 = 0$

$c_{bir}(\mathbb{P}^2, B_0) = 0$

$\mathbb{P}^2 =$ 

$c(\mathbb{P}^2, B_1) = 2 + 1 - 2 = 1.$

$c_{bir}(\mathbb{P}^2, B_1) = 0.$

$\mathbb{P}^2 =$ 

$c(\mathbb{P}^2, B_2) = 2 + 1 - 1 = 2.$

$c_{bir}(\mathbb{P}^2, B_2) = 0.$

**Theorem** (Mauri – M, 24): Let $(X, B)$ be a log CY pair of index one (meaning $K_X + B \sim 0$).
Then $c_{bir}(X, B) = 0$ if and only if

$$(X, B) \simeq_{bir} (\mathbb{P}^n, H_0 + \dots + H_n)$$

**Sketch:** $c_{bir}(X, B) = 0$, then there is a crepant model $(T, B_T)$ with $c(T, B_T) = 0$.

$\xrightarrow{\text{BMSZ16}}$ $T$ is toric, further $K_T + B_T \sim 0$.

& $\lfloor B_T \rfloor$ is toric, so $B_T$ is toric.

$\xrightarrow{\text{toric geom}}$ $B_T = T \setminus \mathbb{G}_m^n$.

$\xrightarrow{\text{toric geom}}$ $(T, B_T) \simeq_{bir} (\mathbb{P}^n, H_0 + \dots + H_{n+1})$ $\square$

**Q:** What is the best toric model $T$ that we can achieve dropping the index one cond?

# Dual complexes:

$E \subseteq X$ SNC divisor on a smooth var.

$E = \bigcup_{i \in I} E_i$. $E_i$ irreducible.

The "dual complex" of $E$, denoted by $\mathcal{D}(E)$ is a CW complex whose cells of dim $k$ correspond to irreducible components of $\bigcap_{j \in I_0} E_j$ with $|I_0| = k+1$.

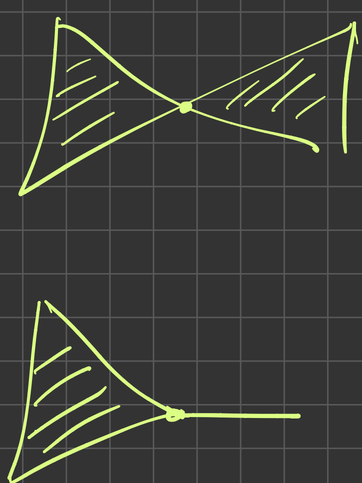
**Definition:** Let $(X, B)$ be a log CY pair & $p : Y \longrightarrow X$ be a log resolution. The "dual complex" of $(X, B)$, denoted by $\mathcal{D}(X, B)$ is $\mathcal{D}(B_Y^{=1})$ where

$$K_Y + B_Y = p^*(K_X + B).$$

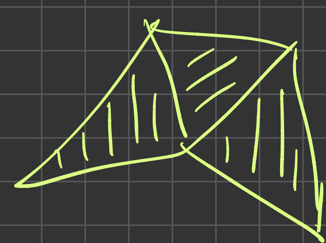
**Theorem (de Fernex – Kollár – Xu, 2012):** For a log CY pair $(X, B)$, the dual complex $\mathcal{D}(X, B)$ is well-defined up to simple homotopy equivalence.

**Theorem (Kollár – Kovács, 2012):** For a log CY pair $(X, B)$ the dual complex $\mathcal{D}(X, B)$ is a pseudo-manifold.

Not pseudo-manifolds:



Pseudo-manifolds:



**Conjecture (Kollár – Xu, 2014):** Let $(X, B)$ be a log CY pair. Then

$$\mathcal{D}(X, B) \simeq_{PL} S^k / G,$$

where $k \leq \dim X - 1$ & $G \leq O(k)$.

**Theorem (Kollár – Xu, 2014):** The conjecture is true in dim 4.

**Definition:** The "coregularity" of a log CY pair $(X,B)$ is $\dim X - \dim \mathcal{D}(X,B) - 1$

$\operatorname{coreg}(X,B) \in \{0,\dots,\dim X\}$.

**Examples:** $(X_n, B_n) := ((\mathbb{P}^1)^n, B_n)$ $n \geq 2$. torus inv boundary.

$$\mathcal{D}(X_n,B_n) = C^{n-1}.$$

$\operatorname{coreg}(X_n,B_n) = 0.$

$c_{bir}(X_n,B_n) = 0.$

$i_n : (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^n$. $\quad i_n^* B_n = B_n$

$([x_1:y_1],\dots,[x_n:y_n]) \longmapsto ([y_1:x_1],\dots,[y_n:x_n]).$

$(X_n, B_n)/i = (Y_n, \Delta_n).$ $\boxed{n \geq 3}$

$\mathcal{D}(Y_n,\Delta_n) = \mathbb{P}^{n-1}_{\mathbb{R}}$. $\quad \pi_1(\mathbb{P}^{n-1}_{\mathbb{R}}) \simeq \mathbb{Z}_2.$

$\operatorname{coreg}(Y_n,\Delta_n) = 0.$

$$c(Y_n,\Delta_n) = \underbrace{\dim Y_n}_{n} + \underbrace{\rho(Y_n)}_{n} - \underbrace{|\Delta_n|}_{n} = n.$$

$0 \leq c_{bir}(Y_n, \Delta_n) \leq n$

By the Corollary (see below) we have $c_{bir}(Y_n,\Delta_n) = n$

**Theorem** (Mauri – M, 24): Let $(X, B)$ be a log CY pair. Then $c_{bir}(X,B) \leq \dim X + \operatorname{coreg}(X,B)$.

**Remark:** If $\operatorname{coreg}(X,B)=0$, then $c_{bir}(X,B) \leq \dim X$.

Pass to a bir model that computes $c_{bir}$.

$$c(X',B') = c_{bir}(X,B)$$

$$c(X',B') \leq \dim X' + \operatorname{coreg}(X',B')$$

$$\cancel{\dim X'} + \rho(X') - |B'| \leq \dim X' + \cancel{\dim X'} - \operatorname{reg}(X',B') - 1$$

$$\operatorname{reg}(X',B') \leq |B'| - \rho(X') + \dim X'.$$

Set $\rho(X')=1$.

$$\operatorname{reg}(X',B') \leq \dim X' + |B'| - 1$$

$B'$ reduced.

✓✓.

**Theorem** (Mauri – M, 24):

Let $(X, B)$ be a log CY pair.

If $c_{bir}(X, B) < \dim X$, then $\mathcal{D}(X, B)$ is union of two collapsible subcomplexes.

links are PL-spheres

Furthermore, if $\mathcal{D}(X, B)$ is smooth of $\dim \neq 4$, then $\mathcal{D}(X, B)$ is either a sphere or a disk (PL).

**Remark:** In dim 4, we get a topological sphere

**Corollary:** Let $(X, B)$ be a CY pair.

If $c_{bir}(X, B) < \dim X$, then $\pi_1(\mathcal{D}(X, B)) = \{1\}$.

**Proof uses:** – Birational Geometry.

– Schoenflies PL Theorem + PL Poincaré.

**Sketch:** Assume $K_X + B \sim 0$ (for simplicity).

What does $c_{bir}(X,B) < \dim X$ mean?

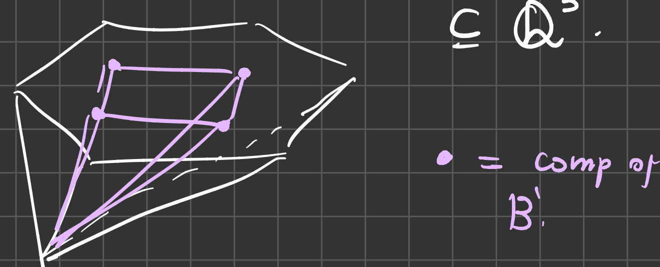
Pick a birational model $(X', B') \simeq_{bir} (X, B)$

with $c_{bir}(X,B) = c(X', B')$.

$$c(X', B') < \dim X'.$$

$$\rho(X') < |B'|$$

Cone of effective div:




argument.

We prove that (in some suitable model), $B'$ properly supports an ample divisor.

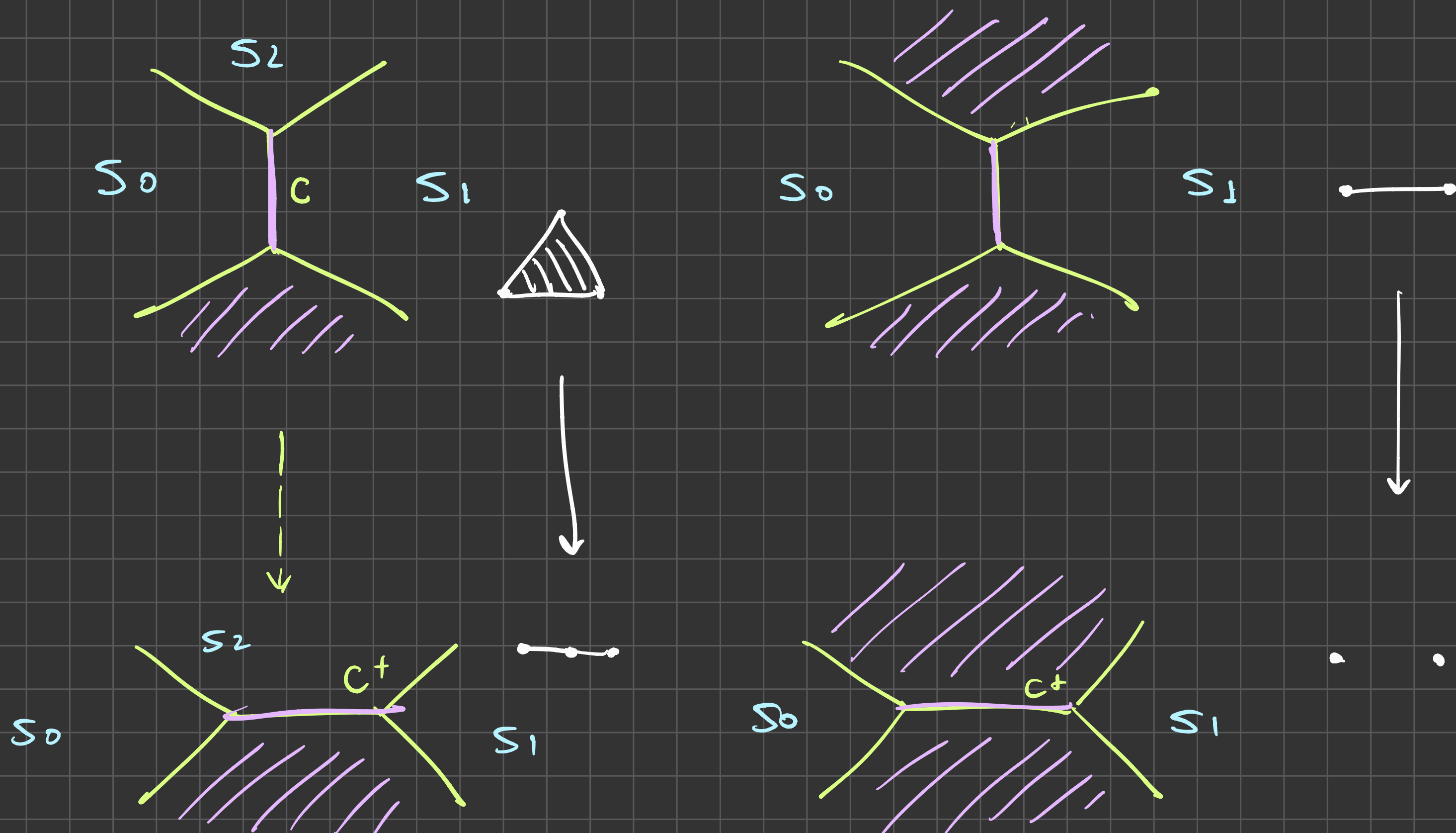
$B' = B'_0 + E$ such that $B'$ supports an ample $A$.

$(K_{X'} + B' - E)$ – MMP with scaling of $A$ that terminates on a Fano variety

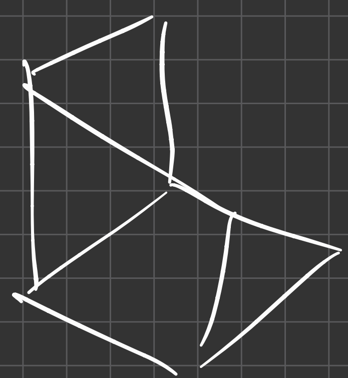
We conclude $\mathcal{D}(X', B' - E)$ is collapsible &

$\mathcal{D}(X', B)$ is collapsible $\square$

Lemma: $(X, B)$ log pair, $X \overset{\varphi}{\dashrightarrow} Y$ is a step of the $(K_X+B)$-MMP & the extremal curve $C$ int some comp of $B$ positively, then $\mathcal{D}(X,B)$ collapses to $\mathcal{D}(Y, B_Y)$ where $B_Y = \varphi_* B$.



Theorem (Logvinov): Let $(X, B)$ be a dlt Fano pair then $\mathcal{D}(X,B)$ is a simplex



$n-1$ in $\mathbb{P}^n$

application of connectedness of lcc.